

Solutions

**DAWSON COLLEGE**  
**DEPARTMENT OF MATHEMATICS**

201-BZF-05

IDENTIFICATION

**CALCULUS-III**

Winter 2011

Time: 3 hours

Instructor: T. Kengatharam and A.Panait

Name:

ID:

**Instructions:**

- Translation and regular dictionaries are permitted.

(1) [5 marks] Find the exact value of the sum

$$= \frac{e^{\pi}}{\pi - e}$$

(2) [5 marks] Find the radius of convergence of the series

$$\sum_{n=1}^{\infty} \frac{n(x+1)^n}{(n^2+1)4^n}$$

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(n+1)(x+1)^{n+1}}{(n^2+2n+2)4^{n+1}} \cdot \frac{n(x+1)^n}{(n^2+1)4^n} \right|$$

use  $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ .)

$$\frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = \sum_{n=0}^{\infty} (-x^2)^n = \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

$$\int \frac{1}{1+x^2} dx = \int \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow \tan^{-1} x + C = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

$$x=0 \Rightarrow C=0 \Rightarrow \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$$

(5) [5 marks] Find the Maclaurin series of  $f(x) = xe^x$  and find the radius of

- (7) [5 marks] Find the curvature  $\kappa = \frac{|\underline{r}' \times \underline{r}''|}{|\underline{r}'|^3}$  for the curve  $\underline{r}(t) = (\sin(3t), \cos(3t), t)$  at the point  $(0, 1, 0)$ .

$$\underline{r}' = (3\cos(3t), -3\sin(3t), 1) \Rightarrow \underline{r}'(0) = (3, 0, 1)$$

$$\underline{r}'' = (-9\sin(3t), -9\cos(3t), 0) \Rightarrow \underline{r}''(0) = (0, -9, 0)$$

$$\underline{r}'(0) \times \underline{r}''(0) = \begin{vmatrix} + & - & + \\ 3 & 0 & 1 \\ 0 & -9 & 0 \end{vmatrix} = (9, 0, -27)$$

$$|\underline{r}'(0) \times \underline{r}''(0)| = \sqrt{9^2 + 0^2 + (-27)^2}$$

$$\kappa(0) = \frac{9\sqrt{10}}{10\sqrt{10}} = \frac{9}{10}$$

- (8) [5 marks] Find the length of the curve  $\underline{r}(t) = (2t, t^2, \frac{1}{3}t^3)$  over the interval  $0 \leq t \leq 1$ .

$$\underline{r}' = (2, 2t, t^2)$$

$$|\underline{r}'| = \sqrt{4 + 4t^2 + t^4} = \sqrt{(t^2 + 2)^2} = t^2 + 2$$

$$\text{Arc length} = \int_0^1 |\underline{r}'| dt = \int_0^1 (t^2 + 2) dt$$

(a)  $x^2 + y^2 = r^2$ 

$$x^2 + y^2 = r^2$$

Parametrization  $\underline{r}(t) = (r \cos t, r \sin t, 0)$ 

$$\underline{r}' = (-r \sin t, r \cos t, 0)$$

$$\underline{r}'' = (-r \cos t, -r \sin t, 0)$$

- (10) [3 marks] Let  $\underline{N}(t)$  be the unit normal vector of a space curve  $\underline{r}(t)$ . Prove that  $\underline{N}(t)$  is orthogonal to  $\underline{N}'(t)$  for all  $t$ .

$$\underline{N} \cdot \underline{N} = 1$$

$$\Rightarrow \underline{N}' \cdot \underline{N} + \underline{N} \cdot \underline{N}' = 0$$

$$\Rightarrow 2(\underline{N}' \cdot \underline{N}) = 0$$

$$\Rightarrow \underline{N}' \cdot \underline{N} = 0$$

$$\Rightarrow \underline{N}' \perp \underline{N}$$

- (11) [5 marks] Evaluate the limit if it exists or show that the limit does not exist.

$$\lim_{(x,y) \rightarrow (0,0)} \frac{6x^3y}{2x^4 + y^4}$$

along  $x$ -axis ( $y=0$ )

$$f(x,0) = \frac{0}{2x^4 + 0} = 0$$

$(x,0) \rightarrow (0,0)$  along  $x$ -axis

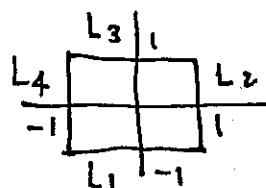
(12) [5 marks] Determine the set of points on which the following function is continuous.



$$(2x, 2y, 2z) = \lambda(1, 1, 1)$$

$$2x = \lambda$$

$$2y = \lambda$$



(15) [7 marks] Find the absolute maximum and absolute minimum of

$$f(x, y) = x^2 + y^2 + x^2y + 4$$

over the rectangle  $R = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$ .

$$f_x = 2x + 2xy = 0 \Rightarrow 2x(1+y) = 0 \Rightarrow x = 0 \text{ or } y = -1$$

$$f_y = 2y + x^2 = 0 \Rightarrow 2y = -x^2$$

$$x = 0 \Rightarrow y = 0$$

$$y = -1 \Rightarrow x = \pm\sqrt{2}$$

$$(0, 0) \in R \Rightarrow \boxed{f(0, 0) = 4}$$

on  $L_1$ :  $y = -1, -1 \leq x \leq 1$

$$f(x, -1) = x^2 + 1 - x^2 + 4 = 5$$

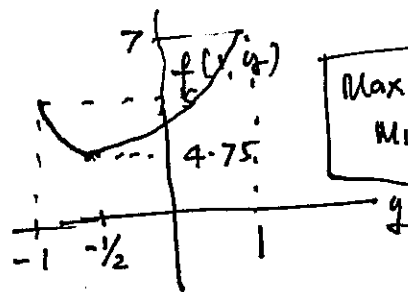
on  $L_2$ :  $x = 1, -1 \leq y \leq 1$

$$f(1, y) = 1 + y^2 + y + 4 = y^2 + y + 5$$

on  $L_3$ :  $(y = 1, -1 \leq x \leq 1)$

$$f(x, 1) = x^2 + 1 + x^2 + 4 = 2x^2 + 5$$

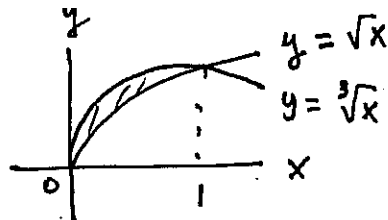
$$f(x, 1) = 2x^2 + 5$$



$$\text{Max } f(1, 1) = 7$$

$$\text{Min } f(1, -\frac{1}{2}) = 4.75$$

$$\text{Min } f(-1, 1) = 7$$



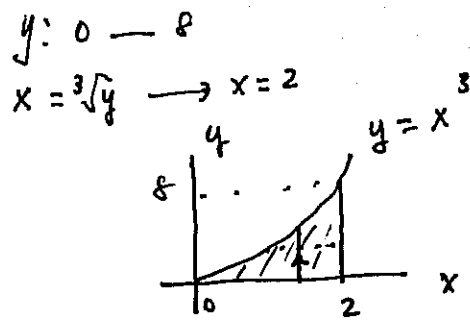
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- (16) [5 marks] Find the volume of the solid under the surface  $z = 2x + y^2$  and above the region bounded by  $x = y^2$  and  $x = y^3$ .

$\int_0^1 \int_{y^3}^{y^2} (2x + y^2) dx dy$

(17) [5 marks] Compute the integral

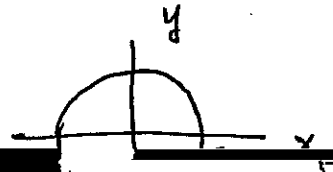
$$\begin{aligned}
 I &= \int_0^2 \int_0^{x^3} e^{x^4} dy dx \\
 &= \int_0^2 x^3 e^{x^4} dx \\
 &= \frac{1}{4} \int_0^{16} e^u du \\
 &= \frac{1}{4} e^u \Big|_0^{16} \\
 &= \frac{1}{4} (e^{16} - 1).
 \end{aligned}$$



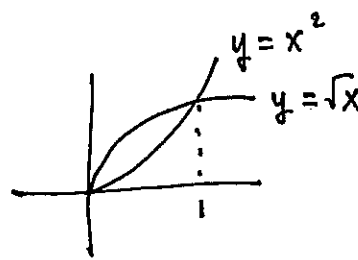
$$\begin{aligned}
 u &= x^4 \\
 \frac{du}{4} &= x^3 dx
 \end{aligned}$$

(18) [5 marks] Find the volume of the solid under the paraboloid  $z = 18 - 2x^2 - 2y^2$  and above the  $xy$ -plane (You may use the polar coordinates).

$$V = \int_0^{2\pi} \int_0^3 (18 - 2r^2) r dr d\theta$$



$$\begin{aligned}
 &= \int_0^{2\pi} \left[ 9r^2 - \frac{2r^4}{4} \right]_0^3 d\theta = \int_0^{2\pi} \left( 81 - \frac{81}{2} \right) d\theta \\
 &= \frac{81}{2} \theta \Big|_0^{2\pi} = 81\pi.
 \end{aligned}$$



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(19) 15 mark Evaluate

planes  $z = 0$  and  $z = x + y$ .

$$I = \int_0^1 \int_0^{\sqrt{x}} \int_0^{x+y} xy \, dz \, dy \, dx = \int_0^1 \int_0^{\sqrt{x}} (x^2 y + xy^2) \, dy \, dx$$