The Riemann Sum and the De[redacted]

We begin our introduction to the Riemann Sum by considering non-neg[redacted] continuous over an interval $[a,b]$. To simplify the explanation and the calc[redacted] $[a,b]$ will be divided into subintervals of equal width, and the sample points wi[redacted] the right endpoints of the subintervals. A more general/rigorous treatment of the Rie[redacted] may be found in the calculus textbook used by Pure and Applied Science students.

Let the non-negative function $y = f(x)$ be continuous over $[a,b]$. We divide $[a,b]$ into n equal subintervals of width

Note: In the following two examples we consider non-negative functions on the interval $[a,b]$. As explained last page, in such cases the definite integral from a to b is the area under the curve from a to b (i.e. the area between the curve and the x-axis). The summation formulas in the appendix will be needed in the solutions of these examples.

Example 1 Use the definition of definite integral to evaluate $\int_0^4 \left(2x^2+3\right) dx$.

We subdivide the interval $[0,4]$ into n equal subintervals of width $x \quad \frac{}{n} \quad \frac{}{n}$

Until now we have only considered non-negative functions on the interval $[a,b]$. In what follows, the function may be negative

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6. – —— ( )

10. $\Delta x = \frac{4}{n} \to x_k = \frac{4k}{n} - 5$ and $f(x_k) = \left(\frac{4k}{n} - 5\right)^2 + 3\left(\frac{4k}{n} - 5\right) + 5 \to f(x_k) = \frac{16k^2}{n^2} - \frac{28k}{n} + 15$

$$f(x_k)\Delta x = \left(\frac{16k^2}{n^2} - \frac{28k}{n} + 15\right)\left(\frac{4}{n}\right) = \frac{64k^2}{n^3} - \frac{112k}{n^2} + \frac{60}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = \frac{64}{n^3}\sum_{k=1}^{n} k^2 - \frac{112}{n^2}\sum_{k=1}^{n} k + \frac{60}{n}\sum_{k\to 1}^{n} 1 = \frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 56\left(1+\frac{1}{n}\right) + 60$$

$$\int_{-5}^{-1}\left(x^2+3x+5\right)dx = \lim_{n\to\infty}\left[\frac{32}{3}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) - 56\left(1+\frac{1}{n}\right) + 60\right] = \frac{32}{3}(1)(2) - 56(1) + 60 = \frac{76}{3}$$

11. $\Delta x = \frac{5}{n} \to x_k = \frac{5k}{n} - 2$ and $f(x_k) = 1 - 5\left(\frac{5k}{n} - 2\right)^2 \to f(x_k) = -\frac{125k^2}{n^2} + \frac{100k}{n} - 19$

$$f(x_k)\Delta x = \left(-\frac{125k^2}{n^2} + \frac{100k}{n} - 19\right)\left(\frac{5}{n}\right) = -\frac{625k^2}{n^3} + \frac{500k}{n^2} - \frac{95}{n}$$

$$\sum_{k=1}^{n} f(x_k)\Delta x = -\frac{625}{n^3}\sum_{k=1}^{n} k^2 + \frac{500}{n^2}\sum_{k=1}^{n} k - \frac{95}{n}\sum_{k=1}^{n} 1 = -\frac{625}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + \frac{500}{2}\left(1+\frac{1}{n}\right) - 95$$

$$\int_{-2}^{3}\left(1-5x^2\right)dx = \lim_{n\to\infty}\left[-\frac{625}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) + 250\left(1+\frac{1}{n}\right) - 95\right] = -\frac{625}{6}(1)(2) + 250(1) - 95 = -\frac{160}{3}$$

12. $\Delta x = \frac{3}{n} \to x_k = \frac{3k}{n} - 1$ and $f(x_k) = \left(\frac{3k}{n} - 1\right)^3 \to f(x_k) = \frac{27k^3}{n^3} - \frac{27k^2}{n^2} + \frac{9k}{n} - 1$

$$f(x_k)\Delta x = \left(\frac{27k^3}{n^3} - \frac{27k^2}{n^2} + \frac{9k}{n} - 1\right)\left(\frac{3}{n}\right) = \frac{81k^3}{n^4} - \frac{81k^2}{n^3} + \frac{27k}{n^2} - \frac{3}{n}$$

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APPENDIX

The following are useful formulas for working with summation notation.

1. $$\sum_{k=1}^{n} c = nc$$

2. $$\sum_{k=1}^{n} c\,a_k = c\sum_{k=1}^{n} a_k$$

3. $$\sum_{k=1}^{n} \left(a_k + b_k \right) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$$

4. $$\sum_{k=1}^{n} \left(a_k - b_k \right) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$$

5. $$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

6. $$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

7. $$\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$$