L'Hôpital's Rule

In this note we will evaluate the limits of some **indeterminate forms** using **L'Hôpital's Rule.**

Indeterminate Forms $\frac{\infty}{\infty}$ and $\frac{0}{0}$

Suppose $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$. Then $\lim_{x \to a} \frac{f(x)}{g(x)}$ may or may not exist and it is called the **indeterminate form of type** $\frac{0}{0}$.

Suppose $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$. Then $\lim_{x \to a} \frac{f(x)}{g(x)}$ may or may not exist and it is called the **indeterminate form of type** $\frac{\infty}{\infty}$.

Note that a can represent a nite real number or $+\infty$ or $-\infty$.

> **L'Hôpital's Rule:** Suppose f and g are di erentiable on an open interval containing a and $g'(x) \neq 0$ on that interval except possibly for a. Also suppose that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Then
>
> $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$
>
> provided that the limit on the right side exists or is $\pm\infty$.

Example 1. Evaluate $\lim_{x \to 2} \frac{5x^3 - 13x^2 + 6x}{4x^2 - 13x + 10}$:

Solution. Since $\lim_{x \to 2} 5x^3 - 13x^2 + 6x = 0$ and $\lim_{x \to 2} 4^2 - 13x + 10 = 0$ we can apply L'Hôpital's Rule. So

$$\lim_{x \to 2} \frac{5x^3 - 13x^2 + 6x}{4x^2 - 13x + 10} = \lim_{x \to 2} \frac{15x^2 - 26x + 6}{8x - 13} = \frac{14}{3}$$

Example 2. Evaluate $\lim_{x \to \infty} \frac{10x + 5}{3x^2 - 7x + 4}$:

Solution. Since $\lim_{x \to \infty} 10x + 5 = \infty$ and $\lim_{x \to 2} 3x^2 - 7x + 4 = \infty$ we can apply L'Hôpital's Rule. So

$$\lim_{x \to \infty} \frac{10x + 5}{3x^2 - 7x + 4} = \lim_{x \to \infty} \frac{10}{6x - 7} = 0$$

Example 3. Evaluate $\lim_{x\to 0} \dfrac{e^x}{1-\cos x}$:

Solution. We can NOT apply L'Hôpital's Rule because $\lim_{x\to 0} e^x = 1$ and $\lim_{x\to 0} 1-\cos x = 0$: Therefore

$$\lim_{x\to 0} \frac{e^x}{1-\cos x} = \infty$$

Example 4. Find $\lim_{x\to 0} \dfrac{\cos x - 1}{e^x - 1}$:

Solution. Since $\lim_{x\to 0} \cos x - 1 = 0$ and $\lim_{x\to 0} e^x - 1 = 0$ we can apply L'Hôpital's Rule. So

$$\lim_{x\to 0} \frac{\cos x - 1}{e^x - 1} = \lim_{x\to 0} \frac{-\sin x}{e^x} = 0$$

Example 5. Evaluate $\lim_{x\to 1^+} \dfrac{7\sqrt{x-1}}{\sin(x-1)}$:

Solution. Since $\lim_{x\to 1^+} 7\sqrt{x-1} = 0$ and $\lim_{x\to 1^+} \sin(x-1) = 0$ we can apply L'Hôpital's Rule. So

$$\lim_{x\to 1^+} \frac{7\sqrt{x-1}}{\sin(x-1)} = \lim_{x\to 1^+} \frac{\frac{7}{2\sqrt{x-1}}}{\cos(x-1)} = \lim_{x\to 1^+} \frac{7}{2\cos(x-1)\sqrt{x-1}} = \infty$$

Example 6. Find $\lim_{x\to\infty} \dfrac{3\ln(5x+3)}{2\ln(x+4)}$:

Solution. Since $\lim_{x\to\infty} 3\ln(5x+3) = \infty$ and $\lim_{x\to\infty} 2\ln(x+4) = \infty$ we can apply L'Hôpital's Rule. So

$$\lim_{x\to\infty} \frac{3\ln(5x+3)}{2\ln(x+4)} = \lim_{x\to\infty} \frac{\frac{15}{(5x+3)}}{\frac{2}{(x+4)}} = \lim_{x\to\infty} \frac{15x+60}{10x+6} = \frac{15}{10} = \frac{3}{2}$$

Example 7. Find $\lim_{x\to 0^+} \dfrac{e^x - 1 - x}{x\sin x}$:

Solution. Since $\lim_{x\to 0^+} e^x - 1 - x = 0$ and $\lim_{x\to 0^+} x\sin x = 0$ we can apply L'Hôpital's Rule. So

$$\lim_{x\to 0^+} \frac{e^x - 1 - x}{x\sin x} = \lim_{x\to 0^+} \frac{e^x - 1}{\sin x + x\cos x}$$

As you see $\lim_{x\to 0^+} e^x - 1 = 0$ and $\lim_{x\to 0^+} \sin x + x\cos x = 0$, so we need to reapply L'Hôpital's Rule:

$$\lim_{x\to 0^+} \frac{e^x - 1 - x}{x\sin x} = \lim_{x\to 0^+} \frac{e^x - 1}{\sin x + x\cos x} = \lim_{x\to 0^+} \frac{e^x}{\cos x + \cos x - x\sin x} = \frac{1}{2}$$

Indeterminate Form $0 \cdot \infty$

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty (\text{or} -\infty)$ then it is not clear what the value of $\lim_{x \to a} f(x)g(x)$, if any, will be. This is called the **indeterminate form of type** $0 \cdot \infty$.

We can convert this type into an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by writing the product fg as a quotient

$$fg = \frac{f}{\frac{1}{g}} \qquad \text{or} \qquad fg = \frac{g}{\frac{1}{f}}$$

Example 8. Evaluate $\lim_{x \to 0^+} x \ln x$:

Solution. Since $\lim_{x \to 0^+} x = 0$ and $\lim_{x \to 0^+} \ln x = -\infty$, the limit is an indeterminate form of type $0 \cdot \infty$. First we convert this product into the following quotient

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}}$$

where the right side is an indeterminate form of $\frac{\infty}{\infty}$. Then using L'Hôpital's Rule we have:

$$\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0^+} -x = 0$$

Example 9. Evaluate $\lim_{x \to \infty} x \tan(1=x)$:

Solution. Since $\lim_{x \to \infty} x = \infty$ and $\lim_{x \to \infty} \tan(1=x) = 0$, the limit is an indeterminate form of type $0 \cdot \infty$. First we convert this product into the following quotient

$$\lim_{x \to \infty} x \tan(1=x) = \lim_{x \to \infty} \frac{\tan(1=x)}{1=x}$$

where the right side is an indeterminate form of $\frac{0}{0}$. Then L'Hôpital's Rule implies that:

$$\lim_{x \to \infty} x \tan(1=x) = \lim_{x \to \infty} \frac{\tan(1=x)}{1=x} = \lim_{x \to \infty} \frac{-1=x^2 \sec^2(1=x)}{-1=x^2} = \lim_{x \to \infty} \sec^2(1=x) = 1$$

Example 10. Evaluate $\lim_{x \to \infty} x^3 e^{-x^2}$:

Solution. It is not di cult to see that the limit is an indeterminate form of type $0 \cdot \infty$. We can easily convert it into the quotient $\lim_{x \to \infty} \frac{x^3}{e^{x^2}}$ that gives an indeterminate form $\frac{\infty}{\infty}$ and then apply L'Hôpital's Rule twice we will have:

$$\lim_{x \to \infty} x^3 e^{-x^2} = \lim_{x \to \infty} \frac{x^3}{e^{x^2}} = \lim_{x \to \infty} \frac{3x^2}{2xe^{x^2}} = \lim_{x \to \infty} \frac{3x}{2e^{x^2}} = \lim_{x \to \infty} \frac{3}{4xe^{x^2}} = 0$$

Indeterminate Form $\infty - \infty$

If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$ then the limit $\lim_{x \to a}[f(x) - g(x)]$ is called the **indeterminate form of type** $\infty - \infty$.

We can convert this type into an indeterminate form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by using a common denominator, or factoring out a common factor or rationalization.

Example 11. Evaluate $\lim_{x \to (\ \ =2)} (\sec x - \tan x)$:

Solution. Since $\lim_{x \to (\ \ =2)} \sec x = \infty$ and $\lim_{x \to (\ \ =2)} \tan x = \infty$, the given limit is an indeterminate form $\infty - \infty$. Here we use a common denominator to convert it into $\frac{0}{0}$ and then we apply L'Hôpital's Rule:

$$\lim_{x \to (\ \ =2)} (\sec x - \tan x) = \lim_{x \to (\ \ =2)} \left(\frac{1}{\cos x} - \frac{\sin x}{\cos x}\right) = \lim_{x \to (\ \ =2)} (\ \text{sin}$$

Exercises. Evaluate the following limits. Use L'Hôpital's Rule where appropriate.

1. $\displaystyle \lim_{x\to 3} \frac{2x^2 - 3x - 9}{x^2 - 2x - 3}$

2. $\displaystyle \lim_{x\to \infty} \frac{4x^3 + x - 3}{x^2 - 5x + 8}$

3. $\displaystyle \lim_{x\to 1} \frac{2x^3 - x^2 - 4x + 3}{3x^3 - 5x^2 + x + 1}$

4. $\displaystyle \lim_{x\to \infty} \frac{6x - 5}{4x^2 + 7x + 9}$

5. $\displaystyle \lim_{x\to 0} \frac{1 - e^x}{2x}$

6. $\displaystyle \lim_{x\to 0} \frac{x^2 + x}{\sin 3x}$

7. $\displaystyle \lim_{x\to 0^+} \frac{\sin x}{1 - \cos x}$

8. $\displaystyle \lim_{x\to 0} \frac{\sin 2x}{\sin x}$

9. $\displaystyle \lim_{x\to 0} \frac{e^x - x - 1}{5x^2}$

10. $\displaystyle \lim_{x\to \infty} \frac{e^{3x}}{\ln x}$

11. $\displaystyle \lim_{x\to 1} \frac{x - 1}{\sqrt{x} - 1}$

12. $\displaystyle \lim_{x\to 0^+} \frac{\sqrt{x}}{1 - \cos x}$

13. $\displaystyle \lim_{x\to \infty} \frac{\sqrt{x - 1}}{4x + 5}$

14. $\displaystyle \lim_{x\to 1} \frac{\ln x}{\sin(x - 1)}$

15. $\displaystyle \lim_{x\to \infty} \frac{\ln(x^2 + 1)}{\sqrt{x}}$

16. $\displaystyle \lim_{x\to 1} \frac{e^{x-1} - 1}{(x - 1)^3}$

17. $\displaystyle \lim_{x\to \infty} \frac{\ln(x - 10)}{\ln(4x + 1)}$

18. $\displaystyle \lim_{x\to 0^+} \frac{\sqrt{x}}{\ln(x + 1)}$

19. $\displaystyle \lim_{x\to \infty} \frac{e^{4x}}{e^{3x} + x}$

20. $\displaystyle \lim_{x\to 0}$ e^{2x} x →0 x 4x 0365 d18 9701 Tf 6.652 3.453 Td [(2). lim