ALGEBRA MODULES (Revised Aug. 2000)

ALGEBRA MODULE ONE

LONG DIVISION OF POLYNOMIALS

[illegible]

x – 2.

$$
\begin{array}{r}
2x^2 + 2x + 7 \\
x - 2 \,\big)\, 2x^3 - 2x^2 + 3x - 5 \\
\underline{2x^3 - 4x^2} \qquad\qquad \\
2x^2 + 3x \qquad \\
\underline{2x^2 - 4x} \qquad \\
7x - 5 \\
\underline{7x - 14} \\
9
\end{array}
$$

[illegible]

Example 2: $p(x) = x^3 - 3x^2 - x + 3$.

The possible rational zeros are the factors of 3 since the leading coefficient is 1. Possible zeros $= \pm 1, \pm 3$. It is easy to verify that $p(1) = 0$, so $x - 1$ is a factor. Using long division:

$$\begin{array}{r|l} & x^2 - 2x - 3 \\ \hline x - 1 & x^3 - 3x^2 - x + 3 \\ & \underline{x^3 - x^2} \\ & \quad -2x^2 - x \\ & \quad \underline{-2x^2 + 2x} \\ & \qquad\quad -3x + 3 \\ & \qquad\quad \underline{-3x + 3} \\ & \qquad\qquad\quad 0 \end{array}$$

$$p(x) = (x - 1)(x - 3)(x + 1)$$

SPECIAL FACTORIZATIONS

PERFECT SQUARES: $(a - b)^2 = a^2 - 2ab + b^2$ $\quad (a + b)^2 = a^2 + 2ab + b^2$

Example 2B: (GROUPING and FACTORING)

$$3x^2 - 5x - 2$$

$$= 3x^2 - 6x + x - 2 \qquad \text{(SPLIT the middle term.)}$$

$$= 3x(x-2) + 1(x-2) \qquad \text{(Grouping and Factoring)}$$

$$= (3x+1)(x-2)$$

FOURTH DEGREE POLYNOMIALS: Fourth degree polynomials in the form $ax^4 + bx^2 + c$ may be factored as quadratics.

Example3: $x^4 - 5x^2 + 4 = (x^2 - 1)(x^2 - 4) = (x-1)(x+1)(x-2)(x+2)$

Example 4: $x^4 + 6x^2 - 7 = (x^2 - 1)(x^2 + 7) = (x-1)(x+1)(x^2 + 7)$

PROBLEMS

A. In the following problems, one of the zeros of the polynomial is given. Use the Factor Theorem and long division to factor the polynomial completely.

ANSWERS

A.

1) $(x-1)(x+1)(x-4)$

2) $(x+3)^2(x-1)$

3) $(x-2)^2(x+1)$

4) $x(x+1)(x-2)(x-8)$

5) $(x-2)(x^2-x+2)$

6) $(x+1)(2x-5)(x+2)$

B.

1) $1+\dfrac{4x+1}{x^2-1}$

2) $x-2+\dfrac{-x+3}{x^2+1}$

3) $x^2-2x+4-\dfrac{6}{x+2}$

4) $x^2-2x+7-\dfrac{9}{x+2}$

5) $x-4+\dfrac{8x+7}{x^3+2x^2+x}$

6) $x+\dfrac{-7x^2+2}{x^3+4x}$

C.

1) $(x-2)(x+2)(x^2+4)$

2) $(x-2)(x^2+2x+4)$

3) $(x-5)(x+3)$

4) $x(x-6)(x+4)$

5) $(3x+2)(2x-1)$

6) $(4x+1)(2x-3)$

7) $(x^2-1)(x^2-9)=(x-1)(x+1)(x-3)(x+3)$

8) $(2x+3)(4x^2-6x+9)$

9) $(x-1)(x+1)(x^2+2)$

10) $(x-2)(x^2+4)$

11) $(x+1)^2(x-2)$

12) $(x+1)(x-2)(x-3)$

13) $(x-3)(x-4)(x+2)$

14) $(x-1)(x-2)(x-6)$

15) $(x+2)^3(x-3)$

ALGEBRA MODULE TWO

EXPONENTS AND RADICALS

REVIEW OF PROPERTIES OF EXPONENTS

Let x and y be real numbers, and let m and n be integers.

1) $x^m x^n = x^{m+n}$ $\qquad$ $2^3 2^2 = 2^{3+2} = 2^5 = 32$

2) $\dfrac{x^m}{x^n} = x^{m-n}$ $\qquad$ $\dfrac{x^5}{x^2} = x^{5-2} = x^3$

3) $x^0 = 1$ $\qquad$ $3^0 = 1$

4) $\dfrac{1}{y^n} = y^{-n}$ $\qquad$ $\dfrac{1}{y^3} = y^{-3}$

6) $\left(x^m\right)^n = x^{mn}$ $\qquad$ $\left(x^2\right)^3 = x^6$

7) $\left(\dfrac{x}{y}\right)^m = \dfrac{x^m}{y^m}$ $\qquad$ $\left(\dfrac{5}{y}\right)^2 = \dfrac{25}{y^2}$

FRACTIONAL EXPONENTS

When we write a fractional exponent such as $x^{m/n}$, we can use property 6) above to evaluate the term.

1) $8^{2/3} = \left(8^{1/3}\right)^2 = 2^2 = 4$

2) $8^{-2/3} = \dfrac{1}{8^{2/3}} = \dfrac{1}{4}$

3) $(-8)^{2/3} = \left(-8^{1/3}\right)^2 = (-2)^2 = 4$

4) $(-8)^{-2/3} = \dfrac{1}{(-8)^{2/3}} = \dfrac{1}{4}$

RADICALS

A. Rewrite the expression to eliminate the radicals in the denominator.

1) $\dfrac{2}{\sqrt{3}} = \dfrac{2}{\sqrt{3}} \cdot \dfrac{\sqrt{3}}{\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$ (this process is called rationalizing the denominator.)

For an expression with a sum or difference involving a radical, such as $x + b\sqrt{y}$ we can eliminate the radical in the denominator by multiplying the numerator and the denominator by the conjugate $x - b\sqrt{y}$. This is also called rationalizing. This process makes use of the difference of squares because

$$\left(x + b\sqrt{y}\right)\left(x - b\sqrt{y}\right) = x^2 - b^2 y$$

2) $\dfrac{4}{3-\sqrt{2}} = \dfrac{4}{3-\sqrt{2}} \cdot \dfrac{3+\sqrt{2}}{3+\sqrt{2}} = \dfrac{4\left(3+\sqrt{2}\right)}{9-2} = \dfrac{4\left(3+\sqrt{2}\right)}{7}$

[illegible]

SOLVED PROBLEMS INVOLVING EXPONENTS

Simplify the following expressions

[illegible]

Solution: Regrouping and combining like terms.

$$\left(7^2 a^2 b^3\right)\left(7^4 a^9 b^2\right) = \left(7^2 7^4\right)\left(a^2 a^9\right)\left(b^3 b^2\right) = 7^6 a^{11} b^5 \quad \text{(adding exponents)}$$

2) $\dfrac{128x^3}{32x^5}$

Solution: We express 128 and 32 as powers of the common base 2.

$$\frac{128x^3}{32x^5} = \frac{2^7 x^3}{2^5 x^5} = 2^2 x^{-2} \quad \text{(subtracting exponents)}$$

$$= \frac{4}{x^2} \qquad \left(x^{-2} = \frac{1}{x^2}\right)$$

$(2x)^{-3}$

[illegible]

6) $3^{2n+1} \cdot 3^{n-2}$

Solution: We add the exponents and combine like terms.

$$3^{2n+1} \cdot 3^{n-2} = 3^{(2n+1)+(n-2)} = 3^{3n-1}$$

7) $\dfrac{2^{4n+1}}{2^{5n-1}}$

Solution: We subtract the exponents and combine like terms.

$$\frac{2^{4n+1}}{2^{5n-1}} = 2^{(4n+1)-(5n-1)} = 2^{(4n-5n)+(1--1)}$$

$$= 2^{-n+2}$$

$$= 2^{2-n} \text{ or } \frac{1}{2^{n-2}}$$

8) $\dfrac{4^{n+3}}{2^{n-1}}$

Solution: Change 4 to 2^2.

Thus

$$\frac{4^{n+3}}{2^{n-1}} = \frac{\left(2^2\right)^{n+3}}{2^{n-1}} = \frac{2^{2n+6}}{2^{n-1}} = 2^{(2n+6)-(n-1)}$$

$$= 2^{(2n-n)+(6--1)}$$

$$= 2^{n+7}$$

9) $\sqrt{50x^4y^8}$

Solution: Use the fact that $\sqrt{abc} = \sqrt{a}\,\sqrt{b}\,\sqrt{c}$

$$\sqrt{50x^4y^8} = \sqrt{50}\,\sqrt{x^4}\,\sqrt{y^8}$$

$$= \sqrt{25}\,\sqrt{2}\,\sqrt{x^4}\,\sqrt{y^8}$$

(NOTE: $\sqrt{x^4} = \sqrt{\left(x^2\right)^2} = x^2$

$\sqrt{y^8} = \sqrt{\left(y^4\right)^2} = y^4$)

$$= 5\sqrt{2}\,x^2y^4$$

$$= \sqrt{2}\left(5x^2y^4\right)$$

10) $\sqrt[3]{128x^4y^7}$

Solution: Use the fact that $\sqrt[3]{abc} = \sqrt[3]{a}\,\sqrt[3]{b}\,\sqrt[3]{c}$.

Also $\sqrt[3]{x^3} = x$ $\quad \sqrt[3]{y^6} = \sqrt[3]{\left(y^2\right)^3} = y^2$ and $128 = 2^7 = 2^6 \cdot 2^1$

So

$$\sqrt[3]{128x^4y^7} = \sqrt[3]{128}\,\sqrt[3]{x^4}\,\sqrt[3]{y^7}$$

$$= \left(\sqrt[3]{2^6}\,\sqrt[3]{2}\right)\left(\sqrt[3]{x^3}\,\sqrt[3]{x}\right)\left(\sqrt[3]{y^6}\,\sqrt[3]{y}\right)$$

$$= \left(2^2xy^2\right)\sqrt[3]{x}\,\sqrt[3]{2}\,\sqrt[3]{y} = \left(4xy^2\right)\sqrt[3]{2xy}$$

Solve for k.

1) $3^{n+1} \cdot 3^k = 3^{n-2}$

Solution: $3^k = \dfrac{3^{n-2}}{3^{n+1}}$ (Divide by 3^{n+1}).

$3^k = 3^{-2-1} = 3^{-3}$. (The n's cancel out.) Comparing exponents we see $k = -3$.

2) $\dfrac{2^{n+3}}{2^{k+1}} = 2^{2n-1}$

Solution: Subtracting exponents on the left side gives

Setting exponents equal and solving for k.

$n - k + 2 = 2n - 1$

$(n+2) - (2n-1) = k$

$-n + 3 = k$

D. Simplify:

1) $(9x^5)(27x^2)$ 2) $\frac{1024x^{-4}y^{-2}}{64x^{-6}y^3}$ 3) $(3x^2y^{-3})^{-2}$

4) $\left(\frac{25x^{-3}}{625y^{-2}}\right)^{-2}$ 5) $\frac{(2x^{-1}y^3)^3(4x^2y^{-3})^2}{(32x^{-5}y^4)^3}$ 6) $\frac{(4x^{-3})^{-2}}{(5y^{-2})^{-4}}$

7) $\frac{5^{2n}\cdot 5^{n+1}}{5^{4n-2}}$ 8) $\frac{3^{n+1}\,9^{n-1}}{(27)^{n+2}}$ 9) $\frac{(32)^{n+1}}{(64)^{n-1}}$ 10) $\sqrt[3]{54x^5y^{13}}$

E. Solve for k. 1) $3^{n+2}\,3^{k-1} = 3^{n-3}$ 2) $\frac{2^{n+1}}{2^{k-1}} = 2^{3n}$

A. 1) 27 2) 9 3) $\frac{1}{16}$ 4) 8 5) $\frac{1}{4}$

ALGEBRA MODULE THREE

RATIONAL FUNCTIONS

[illegible]

The properties for manipulations of fractions apply also to fractions of polynomials which are known as rational functions.

1) ADDITION: $\frac{a}{b}+\frac{c}{d}=\frac{ad+bc}{bd}$

2) MULTIPLICATION: $\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$

3) DIVISION: $\frac{a/b}{c/d}=\frac{a}{b}\cdot\frac{d}{c}=\frac{ad}{bc}$ and also $\frac{a/b}{c}=\frac{a}{b}\cdot\frac{1}{c}=\frac{a}{bc}$

[illegible] d simplify the result as much as possible:

[illegible]

$\frac{1}{} \quad \frac{1}{} \quad \frac{x-(x+h)}{}$

PROBLEMS

Perform the indicated operation and write the answer in factored form where possible:

1) $\frac{x}{}$ $\frac{x-2}{}$ 2) $\frac{2}{}$ $\frac{1}{}$ 3) $\frac{1}{}$ $\frac{4}{}$