

Date: Tuesday Dece

Examiner: S. Shahabi.

Name: _____

- Print your name and st
- All questions are to be
space for your answer,
- No book, notes, graphir
- You are only permitted
- Show all your work and
- This examination book

THIS EXAMINATION E

2

$$\left(\frac{1}{1-t} = 1 + t + t^2 + t^3 + \dots \right) \quad -1 < t < 1$$

(1) [5 pts.] Expand $f(x) = \frac{1}{x^3 - 3x^2 + 3x}$ in powers of $(x-1)$, and determine the interval of convergence of the power series.

Hint: the denominator is equal to $1 - (1-x)^3$.

$$f(x) = \frac{1}{1 - (1-x)^3} = 1 + (1-x)^3 + (1-x)^6 + (1-x)^9 + \dots$$

$$= 1 - (x-1)^3 + (x-1)^6 - (x-1)^9 + \dots$$

$$\text{It is convergent} \iff -1 < (1-x)^3 < 1$$

$$\iff -1 < 1-x < 1$$

$$\iff -1 < x-1 < 1$$

$$\text{Hence: } \begin{cases} R = 1, \text{ radius} \\ I =]0, 2[, \text{ interval} \end{cases}$$

(2) [5 pts.] Find the exact value of $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n \sqrt{3}(2n+1)}$.

Recall: $\arctan(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} x^{2n-1}$.

$$L = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n \cdot \sqrt{3}(2n+1)} \stackrel{m=n+1}{=} \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{3^{m-1} \cdot 3^{\frac{1}{2}}(2m-1)}$$

$$= \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{2m-1} \cdot \left(\frac{1}{\sqrt{3}}\right)^{2m-1} \stackrel{(*)}{=} \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

[Also note:

$$\arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} - \arctan(\sqrt{3})$$

$$= \frac{\pi}{6} - \frac{\pi}{3} = -\frac{\pi}{6}$$

using:

$$\left(\arctan(x) + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} \right) \quad x > 0$$

(3) [4 pts.] If $-1 \leq a < 0$, show $\left| \sum_{n=1}^{\infty} \frac{a^n}{n^2+1} - \sum_{n=1}^N \frac{a^n}{n^2+1} \right| < \frac{1}{(N+1)^2}$. Hint: a^n

Recall: for alternating series $S = \sum_1^{\infty}$
 have $|S - S_N| < b_{N+1}$.

Applying this inequality, we get

$$\left| \sum_1^{\infty} \frac{(-1)^n (-a)^n}{n^2+1} - \sum_1^N \frac{(-1)^n (-a)^n}{n^2+1} \right| < \frac{(-a)^{N+1}}{(N+1)^2}$$

$$(0 < -a < 1)$$

(4) [6 pts.] Find the arc-length parametrization of the curve $\vec{r}(t) = \left\langle \frac{2t}{t^2+1}, \pi, \frac{t^2}{t^2+1} \right\rangle$

$$s(t) = \int_0^t \|\vec{r}'(u)\| du \quad \vec{r}'(u) = \left\langle \frac{2}{1+u^2}, 0, \frac{-2u}{1+u^2} \right\rangle$$

$$= \int_0^t \frac{2}{1+u^2} du \quad \|\vec{r}'(u)\| = \frac{2}{1+u^2}$$

$$= 2 \left[\arctan(u) \right]_0^t$$

$$= 2 \arctan(t) \quad (t \geq 0)$$

$$\rightarrow \arctan(t) = \frac{1}{2}s \rightarrow t = \tan\left(\frac{1}{2}s\right)$$

$$\rightarrow \vec{r} = \vec{r}(s) = \left\langle \frac{2 \tan(\frac{1}{2}s)}{1 + \tan^2(\frac{1}{2}s)}, \pi, \frac{\tan^2(\frac{1}{2}s)}{1 + \tan^2(\frac{1}{2}s)} \right\rangle$$

$$= \langle \sin(s), \pi, -\cos(s) \rangle. \quad (s: tl)$$

Rmk: This is a circle (of radius 1) in a plane pa

Sp

$\left(\frac{R_m}{a} \right)$

Exp
(x, y)

$\sqrt{|s|}$

$-\frac{1}{2}$

Send
theo

$\frac{e}{c}$

or

or

$$\frac{z}{i\pi z}$$

$$\Rightarrow \frac{z^2}{z}$$

$$\Rightarrow$$

• Ove

fra

(\therefore f

• Crit

lins

Sin

(11) [5 pts.] Find $\frac{\partial z}{\partial v}$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial v}$$

$$= (2x + y) \cos(x)$$

(12) [6 pts.] Compute

$$I = \int_1^2 \left(\int_1^3 \frac{1}{1+y} \right)$$

$$= \int_1^2 (\ln(y+4))$$

$$= \left[(y+4) (\ln(y+4)) - (y+4) \right]_1^2$$

$$= \left\{ 6 (\ln(6)) - 6 \right\} - \left\{ 5 (\ln(5)) - 5 \right\}$$

$$= 6 \ln(6) - 5 \ln(5) - 1$$

$$= \ln \left(\frac{6^6 \cdot 3^3}{5^5 \cdot 4^4} \right)$$

(13) [6 pts.] Compute the integral $\iint_D xy\sqrt{x^2}$

$$I = \int_0^{\pi/2} \left(\int_0^1 r \cos(\theta) r \sin(\theta) \right)$$

$$= \int_0^{\pi/2} \sin(\theta) \cos(\theta) \left[\frac{1}{5} r^5 \right]$$

$$= \frac{1}{10} \int_0^{\pi/2} \sin(2\theta) d\theta$$

$$= \frac{1}{10} \left(-\frac{1}{2} \cos(2\theta) \right)_0^{\pi/2}$$

$$= \boxed{\frac{1}{10}}$$

(14) [8 pts.] Consider the solid S (in \mathbb{R}^3) bounded by

(i) Setup a *double-iterated* integral which gives the volume of S .

$$z = f(x, y) = \frac{1}{3}(6 - x - 2y)$$

$$R: 0 \leq x \leq 2, \quad x \leq y \leq \frac{1}{2}(6 - x)$$

$$V = \int_0^2 \left(\int_x^{\frac{1}{2}(6-x)} \frac{1}{3}(6 - x - 2y) dy \right) dx$$

(ii) Setup a *triple-iterated* integral which gives the volume of S .

$$V = \iiint_D 1 dV = \int_0^2 \left(\int_x^{\frac{1}{2}(6-x)} \int_0^{\frac{1}{3}(6-x-2y)} dz \right) dy dx$$

(15) [5 pt]
 $x^2 +$

$$A = \int_{\pi/6}^{5\pi/6} (\dots)$$

$$\begin{cases} x^2 + y^2 = 1 \\ x^2 + (y-1)^2 = 1 \end{cases}$$

(16) [12 pt]
Cylind
cone
 $\rho \cos$
(i) V

(ii) V

(iii)